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The flow of water through a dam of fairly arbitrary shape is examined in the limit of large frictional drag on the flow. The relation of the problem to the one-phase Stefan problem and a problem of anodic smoothing is explored. Monotonicity and uniqueness results for the solution of the time-dependent problem are obtained. Numerical solution of the time-dependent problem is considered. Algorithms to treat the steady state problem are studied in detail.

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RELATION OF THE ONE-PHASE STEFAN PROBLEM TO THE SEEPAGE OF LIQUIDS AND ELECTROCHEMICAL MACHINING

Joel C. W. Rogers

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RELATION OF THE ONE-PHASE STEFAN PROBLEM TO THE SEEPAGE OF LIQUIDS AND ELECTROCHEMICAL MACHINING

Joel C. W. Rogers

1. Formulation of the Problems

For several reasons, I have recently been interested in the problem of seepage of liquids through a dam. First, my experience with algorithms for computing generalized solutions of other hydrodynamic free boundary problems (water waves, Ref. 9) has naturally led me to the question as to the applicability and usefulness of such methods for the dam problem. Second, the "Baiocchi transformation" which has been used effectively to solve some free boundary problems of seeping liquids is reminiscent of another transformation, relating the steady state of a one-phase Stefan problem to a time-dependent free boundary problem arising in the theory of anodic smoothing (Ref. 8), and the possibility that the two transformations might have a common origin has intrigued me. A third reason is that, in the second announcement of this intensive seminar, I was billed as discussing "numerical methods" and the dam problem, which is related through the Baiocchi transformation to other free boundary problems in whose numerical solution I

have participated (Refs. 2-5), seemed like an apt problem to discuss. Finally, there is no better forum in which I can discuss this subject than this one in Pavia, where so much work on the dam problem has been done.

The format of this paper is as follows: In the remainder of this introductory section we will formulate, in a cursory fashion, the free boundary problem associated with the seepage of a liquid, as well as variants of the one-phase Stefan problem; for these we will also provide algorithms which have been used to effect their numerical solution. The next section will contain a discussion of the time-dependent porous flow problem, its relation to the time-dependent anodic smoothing problem, and some comments about the numerical solution of the seepage problem. In a third section we examine the *solution of* the steady state porous flow problem for rather general dam shapes in greater detail; we propose an algorithm for the solution of the problem and make some tentative remarks pertinent to the question of error estimates for the approximate solution generated by the algorithm. A final section indicates briefly a possible generalization of this work. Owing to time limitations, the results I report in this paper are not as complete as I would like them to be, and I apologize in advance for this incompleteness. Nevertheless, if the perspective offered in this paper usefully complements other work on the dam problem and thereby contributes to its solution, I will consider the paper to have served its purpose.

By the one-phase Stefan problem in a bounded region (Refs. 5, 8), we will mean the problem of finding the solution of the equation

$$u_t = \Delta f(u) \quad , \quad x \in D \quad , \quad 0 < t < \infty \quad , \qquad (1.1a)$$

subject to boundary conditions

$$Lf(u) = 0 \quad , \quad x \in \partial D \quad , \quad 0 < t < \infty \quad , \qquad (1.1b)$$

and initial conditions

$$u(x,0) = u_0 \quad , \quad x \in D \quad , \qquad (1.1c)$$

where L is an appropriate linear operator and the function f is given by

$$f(u) = \max(u - 1,0) \quad . \qquad (1.2)$$

Under most conditions of interest, the following algorithm suffices for the calculation of u(x,t) (Refs. 4, 5, 8):

$$u(n\tau) \sim u^n \quad , \qquad (1.3a)$$

$$u^0 = u_0 \quad , \qquad (1.3b)$$

$$u^{n+1} = u^n + (S_1(\tau) - 1)f(u^n) \quad , \qquad (1.3c)$$

where the operator $S_1(\tau)$ is defined by

$$\xi(\tau) \equiv S_1(\tau)\xi_0 \qquad (1.4a)$$

when ξ satisfies

$$\xi_t = \Delta\xi \quad , \quad x \in D \quad , \quad 0 < t < \tau \qquad (1.4b)$$

$$L\xi = 0 \quad , \quad x \in \partial D \quad , \quad 0 < t < \tau \quad , \qquad (1.4c)$$

$$\xi(0) = \xi_0 \quad , \quad x \in D \quad . \qquad (1.4d)$$

We shall refer to the algorithm (1.3) as algorithm I, A steady state $\overline{u}$ exists, and algorithm I may be used to calculate $\overline{u}$.

A variant on the one-phase Stefan problem which has a more direct connection to the dam problem is the following set of conditions satisfied by the function

$$v(x,t) = \int_0^t f(u(x,t'))dt' \qquad (1.5)$$

when

$$u_0(x) \geq 1 \text{ for } x \in \Omega(0) \subset D \quad , \qquad (1.6a)$$

$$u_0(x) \equiv 0 \text{ for } x \in D - \Omega(0) \quad : \qquad (1.6b)$$

$$v_t = \Delta v + u_0 - 1 \quad , \quad x \in \Omega(t) \quad , \quad 0 < t < \infty \quad , \qquad (1.7a)$$

$$v|_{\partial\Omega(t)-\partial D} = 0 \quad , \quad 0 < t < \infty \quad , \qquad (1.7b)$$

$$\nabla v|_{\partial\Omega(t)-\partial D} = 0 \quad , \quad 0 < t < \infty \quad , \qquad (1.7c)$$

$$Lv = 0 \quad , \quad x \in \partial D \quad , \quad 0 < t < \infty \quad , \qquad (1.7d)$$

$$v(x,0) = 0 \quad , \quad x \in D \quad , \qquad (1.7e)$$

which has the property that

$$\text{supp } v(x,t) = \Omega(t) \subset D \quad , \qquad (1.8a)$$

$$\Omega(t_1) \supset \Omega(t_2) \text{ for } t_1 \geq t_2 \qquad (1.8b)$$

$$v \text{ and } \nabla v \text{ are continuous in } D \quad . \qquad (1.8c)$$

Obviously, we may use (1.5) and algorithm I to calculate v. However, problem (1.7) is of a type which occurs in the theory of oxygen transport in tissue, and the following algorithm may be used to calculate v(x,t) (Ref. 3):

$$v(n\tau) \sim v^n \quad , \qquad (1.9a)$$

$$v^0 = 0 \quad , \qquad (1.9b)$$

$$v^{n+1} = \max(S_2(\tau)v^n - \tau, 0) \quad , \qquad (1.9c)$$

where $S_2(\tau)$ is defined by

$$\xi(\tau) \equiv S_2(\tau)\xi_0 \qquad (1.10a)$$

when ξ satisfies

$$\xi_t = \Delta\xi + u_0 \quad , \quad x \in D \quad , \quad 0 < t < \tau \quad , \tag{1.10b}$$

$$L\xi = 0 \quad , \quad x \in \partial D \quad , \quad 0 < t < \tau \quad , \tag{1.10c}$$

$$\xi(0) = \xi_0 \quad , \quad x \in D \quad . \tag{1.10d}$$

We refer to the algorithm (1.9) as algorithm II. For this algorithm an explicit error estimate is available (Ref. 3). It is

$$\sup_{x \in D} |v(x,n\tau) - v^n(x)| \leq \tau \quad . \tag{1.11}$$

In a system of coordinates $(x,\tilde{t})$ in which the gravitational acceleration is unity, an irrotational and incompressible flow in a porous medium, assumed to be an open set D, is given by a velocity potential $\tilde{\phi}$ satisfying the equations

$$\tilde{\phi}_{\tilde{t}} = - z - p - \alpha\tilde{\phi} - \tfrac{1}{2}(\nabla\tilde{\phi})^2 \quad , \quad x \in \tilde{\Omega}(\tilde{t}) \subset D \quad , \quad 0 < \tilde{t} < \infty \quad , \tag{1.12a}$$

$$\Delta\tilde{\phi} = 0 \quad , \quad x \in \tilde{\Omega}(\tilde{t}) \quad , \quad 0 < \tilde{t} < \infty \quad , \tag{1.12b}$$

$$p \equiv 0 \text{ in } D - \tilde{\Omega}(\tilde{t}) \quad , \tag{1.12c}$$

where p is the pressure and $\alpha\tilde{\phi}$ is the potential for a frictional drag force on the flow. In the dam problem in N dimensions,

$$D \subset R^{N-1}x(0,\infty) \quad , \tag{1.13a}$$

and with ∂D decomposed as

$$\partial D = \partial D_0 \cup \partial D_R \cup \partial D_I \quad , \tag{1.13b}$$

the following boundary conditions are imposed:

$$p = 0 \text{ on } \partial D_0 \quad , \tag{1.13c}$$

$$p = z_R - z = \text{hydrostatic pressure,}$$

$$\text{or more generally } p = p_R > 0 \text{ on } \partial D_R, \tag{1.13d}$$

$$n \cdot \nabla\tilde{\phi} \text{ is prescribed on } \partial D_I \quad . \tag{1.13e}$$

Making the further decomposition $\partial\mathcal{D}_I = \partial\mathcal{D}_{I_1} \cup \partial\mathcal{D}_{I_2}$ where $\partial\mathcal{D}_{I_2} = \text{supp}\, \frac{\partial\phi}{\partial n}\Big|_{\partial\mathcal{D}_I}$, we require that

$$\partial\tilde{\Omega}(\tilde{t}) \supset \partial\mathcal{D}_R \cup \partial\mathcal{D}_{I_2} \quad . \qquad (1.13f)$$

The only case of physical importance is the case

$$p \geq 0 \text{ throughout } \mathcal{D} \quad . \qquad (1.13g)$$

This constraint should be viewed as a restriction on the types of inflows and outflows $\frac{\partial\tilde{\phi}}{\partial n}$ which may be prescribed on $\partial\mathcal{D}_I$ in (1.13e). A case frequently considered is that for which $\partial\mathcal{D}_I$ is impermeable: $\partial\mathcal{D}_{I_2} = \emptyset$. In the literature, many authors have made the further assumption that $\partial\mathcal{D}_I = B_0$, where

$$B_{z_0} \equiv \{(x,z) \in C\ell(\mathcal{D}) | z = z_0\} \quad . \qquad (1.14)$$

$\partial\mathcal{D}_R$ is that portion of $\partial\mathcal{D}$ in contact with a reservoir of prescribed surface height z_R, and $z < z_R$ on $\partial\mathcal{D}_R$. We assume that p, given on $\partial\mathcal{D}_0 \cup \partial\mathcal{D}_R$ by (1.13c) and (1.13d), has bounded derivatives on $\partial\mathcal{D}_0 \cup \partial\mathcal{D}_R$. The boundary of $\tilde{\Omega}(\tilde{t})$ is decomposed as

$$\partial\tilde{\Omega} = \partial\tilde{\Omega} \cap \partial\mathcal{D} \cup \partial\tilde{\Omega}_f \quad . \qquad (1.15)$$

Boundary conditions on $\partial\tilde{\Omega}_f$ are

$$p = 0 \quad , \quad x \in \partial\tilde{\Omega}_f(\tilde{t}) \quad , \quad 0 < \tilde{t} < \infty \quad , \qquad (1.16)$$

and points $x(\tilde{t})$ on $\partial\tilde{\Omega}_f(\tilde{t})$ move with the velocity

$$\frac{dx}{d\tilde{t}} = \nabla\tilde{\phi} \quad . \qquad (1.17)$$

An initial value problem for the flow in $\mathcal{D}$ would have as its object the determination of a solution $(\tilde{\phi}(\tilde{t}),\tilde{\Omega}(\tilde{t}))$ of (1.12)-(1.17) from the initial data $(\tilde{\phi}(0),\tilde{\Omega}(0))$. However, this problem will not generally possess a unique solution. The reason is that,

for an irrotational incompressible flow in a region D, one is free to specify not only the velocity potential where the density is nonzero, but also the density where the flow enters D. In our problem the velocity field is given by $\tilde{\phi}$ where it is specified, and the density (which essentially takes on only the values 0 and 1) is given by specifying the region $\tilde{\Omega}(\tilde{t})$ occupied by fluid. Accordingly, given a flow region $\tilde{\Omega}(\tilde{t})$, we are free to allow fluid to enter anywhere on $\partial D_0 - \partial\tilde{\Omega}(\tilde{t})$, thereby augmenting the flow region. The restriction (1.12c) only limits the types of such inflows to masses of fluid in free fall in a frictional medium. Such a flow can be found by solving (1.12a) with $p \equiv 0$, yielding

$$\tilde{\phi}_0(x;\tilde{t}) = e^{-\alpha(\tilde{t}-\tilde{t}_0)}\gamma \cdot x - 1/\alpha(1 - e^{-\alpha(\tilde{t}-\tilde{t}_0)})z + \text{constant} \tag{1.18a}$$

in a region $\tilde{\Omega}_0(\tilde{t})$ moving with the fluid with the spatially constant velocity

$$\nabla\tilde{\phi}_0 = \gamma e^{-\alpha(\tilde{t}-\tilde{t}_0)} - \frac{\vec{k}}{\alpha}(1 - e^{-\alpha(\tilde{t}-\tilde{t}_0)}) \quad , \tag{1.18b}$$

and making its appearance at time $\tilde{t}_0$ in D at points of $\partial D_0 - \partial\tilde{\Omega}(\tilde{t}_0)$ for which $n \cdot \gamma < 0$. Denote the augmented region by

$$\tilde{\Omega}^+(t) \equiv \tilde{\Omega}(\tilde{t}) \cup \tilde{\Omega}_0(\tilde{t}) \quad . \tag{1.19}$$

It follows from our discussion that the velocity potential $\tilde{\phi}^+(\tilde{t})$ defined in $\tilde{\Omega}^+(\tilde{t})$ will have the property that $\frac{\partial\tilde{\phi}^+(\tilde{t}_0)}{\partial n} < 0$ for some point on $\partial\tilde{\Omega}^+(\tilde{t}_0) \cap \partial D_0$. We can make the solution of (1.12)-(1.17), subject to given initial conditions, unique, by requiring, for a given decomposition

$$\partial D_0 = \partial D_1 \cup \partial D_2 \tag{1.20}$$

of ∂D_0, that for $\tilde{t} > 0$

(1.21a)

$$\frac{\partial\tilde{\phi}(\tilde{t})}{\partial n} \geq 0 \quad , \quad x \in \partial\tilde{\Omega}(\tilde{t}) \cap \partial D_1 \tag{1.21a}$$

and

$$\partial\tilde{\Omega}(\tilde{t}) \supset \partial D_2 \quad . \tag{1.21b}$$

In flow through a porous medium, the case of interest is that in which the frictional drag coefficient α is so large that the dependent variables describing the flow change insignificantly over the time $1/\alpha$. We may think of $1/\alpha$ as a "relaxation time" during which the flow assumes the asymptotic form associated with the limit $\alpha \to \infty$. The actual asymptotic parameter is $\alpha^2 a$, where a is a characteristic dimension of the dam D.

Because of the large drag force, it is clear that we will be dealing with a very slow flow, and that in the limit as $\alpha \to \infty$ there will have to be some balance in (1.12a) between the drag potential $\alpha\tilde{\phi}$ and the potential of the gravitational and pressure forces, $z + p$. Thus, we shall work with the dependent variable

$$\phi = \alpha\tilde{\phi} \quad . \tag{1.22}$$

Since the relaxation time is $O(1/\alpha)$, we may investigate the phenomenon of relaxation by working with the independent variable

$$\hat{t} = \alpha\tilde{t} \quad . \tag{1.23}$$

Then at a point on $\partial\tilde{\Omega}_f$, using (1.16), (1.17), and (1.12a), we get for the rate of change with time of ϕ for a point moving with the fluid,

$$\frac{d\phi}{d\hat{t}} = - z - \phi + \frac{1}{2\alpha^2}(\nabla\phi)^2 \xrightarrow[\alpha\to\infty]{} - z - \phi \tag{1.24}$$

while from (1.17)

$$\frac{dz}{d\hat{t}} = \frac{1}{\alpha^2}\phi_z \xrightarrow[\alpha\to\infty]{} 0 \quad . \tag{1.25}$$

Accordingly, the asymptotic solution of (1.24) and (1.25) when $\phi(0)$ and $z(0)$ are given, for times $\hat{t} = O(1)$, is

$$\phi(\hat{t}) = -z(0) + e^{-\hat{t}}(\phi(0) + z(0)) \quad . \tag{1.26}$$

It follows from (1.17) and (1.22) that the primary features of the flow will develop, and in particular a steady state will be approached, in times $\tilde{t} = O(\alpha)$. Consequently, we shall study the flow with the independent variable

$$t = \tilde{t}/\alpha \quad . \tag{1.27}$$

From (1.24)-(1.26) we infer that on $\partial\Omega_f(t) \equiv \partial\tilde{\Omega}_f(t\alpha)$,

$$\phi = -z + O(1/\alpha^2) \quad , \tag{1.28}$$

and from

$$\frac{dx}{dt} = \nabla\phi \tag{1.29}$$

we see that after a time $t = O(1)$, the error in the location of the free boundary will be $O(1/\alpha^2)$ if we use instead the boundary condition

$$\phi = -z \quad , \quad x \in \partial\Omega_f(t) \quad , \quad 0 < t < \infty. \tag{1.30}$$

As $\alpha \to \infty$, we can use (1.12a), (1.12b) and (1.22) to get

$$p = -\phi - z \quad , \quad x \in \Omega(t) \quad , \quad 0 < t < \infty \quad , \tag{1.31a}$$

$$\Delta p = 0 \quad , \quad x \in \Omega(t) \quad , \quad 0 < t < \infty \quad , \tag{1.31b}$$

$$p = 0 \text{ on } \partial D_0 \quad , \tag{1.31c}$$

$$p = p_R > 0 \text{ on } \partial D_R \quad , \tag{1.31d}$$

$$n \cdot \nabla p \text{ prescribed on } \partial D_I \cap \partial\Omega(t) \quad , \tag{1.31e}$$

$$p = 0 \text{ on } \partial\Omega_f(t) \quad , \tag{1.31f}$$

and $\partial\Omega_f$ moves with the velocity

$$-\nabla p - \vec{k} \quad . \tag{1.31g}$$

As before, we restrict our attention to problems for which

$$p \geq 0 \quad , \quad x \in D \quad , \quad 0 < t < \infty \quad , \tag{1.31h}$$

and we require

$$\partial\Omega(t) \supset \partial D_R \cup \partial D_{I_2} \quad . \tag{1.31i}$$

Finally, to insure uniqueness, we use the conditions (1.21) in the form

$$-\frac{\partial p}{\partial n} \geq \vec{k} \cdot n \quad , \quad x \in \partial\Omega(t) \cap \partial D_1 \quad , \quad 0 < t < \infty \quad , \tag{1.31j}$$

$$\partial\Omega(t) \supset \partial D_2 \quad , \quad 0 < t < \infty \quad . \tag{1.31k}$$

For the rest of this paper, we will consider (1.31) to be the time-dependent version of the dam problem. This free boundary problem is very similar to the problem for the electrostatic potential in the theory of anodic smoothing (Ref. 8), and in fact the two problems would be identical, were it not for the term $-\vec{k}$ in (1.31g), and the different boundary conditions imposed for p on the "cathode" surface $\partial D_R \cup \partial D_I$.

Remark 1.1: If we consider the evolution of a front in an initially dry porous medium abutting reservoirs, it will follow from (1.31g) that the initial velocity of the front is unbounded. This reflects an error in the approximation (1.30), which only holds after a time $\tilde{t} \approx O(1/\alpha)$, and also an error in using the hydrostatic assumption (1.31d) for the reservoirs, as this breaks down during the initial phase of the flow.

In the beginning of the introduction to this paper, we mentioned the possible applicability of algorithms for other hydrodynamic free boundary problems to the solution of the time-dependent dam problem. Essentially in those more general algorithms the free boundary, the motion of which is given by (1.29), is determined by looking for contours ρ = constant where ρ satisfies an equation like

$$\rho_t + \nabla\phi \cdot \nabla\rho = 0$$

and ρ has a jump at the free boundary. The more general hydrodynamic algorithms do not appear to have any utility for the solution of the dam problem.

2. Comments on the Time-Dependent Dam Problem

A monotonicity result can be deduced immediately for solutions of the dam problem (1.31). We formalize it as a lemma.

Lemma 2.1: Suppose we have two solutions p_1 and p_2 of (1.31) for which $p_1|_{\partial D_R \cup \partial D_0} \geq p_2|_{\partial D_R \cup \partial D_0}$, $\frac{\partial}{\partial n} p_1|_{\partial D_I} \geq \frac{\partial}{\partial n} p_2|_{\partial D_I}$, and $\Omega_1(0) \supset \Omega_2(0)$. Then $\Omega_1(t) \supset \Omega_2(t)$.

Proof: If for some t, we have $\Omega_1(t) \not\supset \Omega_2(t)$, then at some $t_1 < t$ a point of $\partial\Omega_{f_2}(t_1)$ first passed through $\partial\Omega_{f_1}(t_1)$. Then $\Omega_1(t_1) \supset \Omega_2(t_1)$ and $\exists$ a point $P \in \partial\Omega_{f_1}(t_1) \cap \partial\Omega_{f_2}(t_1)$ for which

$$- \frac{\partial p_1}{\partial n}(P,t_1) < - \frac{\partial p_2}{\partial n}(P,t_1) \quad .$$

But $\Delta(p_1 - p_2) = 0$ in $\Omega_2(t_1)$, $p_1 - p_2 \geq 0$ on $\partial\Omega_2(t_1) \cap (\partial D_0 \cup \partial D_R)$, $\frac{\partial}{\partial n}(p_1 - p_2) \geq$ on $\partial\Omega_2(t_1) \cap \partial D_I$, $p_1 - p_2 \geq 0$ on $\partial\Omega_{f_2}(t_1)$, and $(p_1 - p_2)(P,t_1) = 0$. Hence we get

$$- \frac{\partial p_1}{\partial n}(P,t_1) \geq - \frac{\partial p_2}{\partial n}(P,t_1) \quad ,$$

giving a contradiction and proving the lemma.

It follows right away that if the prescribed boundary data are time-dependent and if $\Omega(t)$ is increasing (decreasing) at any time t_0, that is, $\Omega(t_0 + \delta) \supset \Omega(t_0)(\Omega(t_0 + \delta) \subset \Omega(t_0))$ for some $\delta > 0$, then $\Omega(t + \delta) \supset \Omega(t)(\Omega(t + \delta) \subset \Omega(t))$ for all $t \geq t_0$. By picking regions $\Omega(0)$ for which one can determine *a priori* that $- \frac{\partial p}{\partial n} - \vec{k} \cdot n \geq 0$ (≤ 0) everywhere on $\partial\Omega_f(0)$, one can then establish the existence of a steady state solution, assuming that a solution to the time-dependent problem exists.

The following stability result is also easily deduced.

Lemma 2.2: Suppose we have two solutions p_1 and p_2 of (1.31) with the same boundary data prescribed on ∂D. Then

$$\frac{d}{dt}|(\Omega_1(t) - \Omega_2(t)) \cup (\Omega_2(t) - \Omega_1(t))| \leq 0 \quad . \quad (2.1)$$

Proof: Without loss of generality, we can have $\Omega_1(0) \supset \Omega_2(0)$, and all we need to verify is that $\frac{d}{dt}|\Omega_1(t) - \Omega_2(t)|\Big|_{t=0} \leq 0$. From (1.31g),

$$\frac{d}{dt}|\Omega_1(t) - \Omega_2(t)|\Big|_{t=0} = \int_{\partial\Omega_{f_1}(0)} -\left(\frac{\partial p_1}{\partial n} - \vec{k}\cdot n\right)dS$$

$$+ \int_{\partial\Omega_{f_2}(0)}\left(\frac{\partial p_2}{\partial n} + \vec{k}\cdot n\right)dS$$

$$= \int_{\partial\Omega_1(0)\cap\partial\mathcal{D}}\left(\frac{\partial p_1}{\partial n} + \vec{k}\cdot n\right)dS$$

$$- \int_{\partial\Omega_2(0)\cap\partial\mathcal{D}}\left(\frac{\partial p_2}{\partial n} + \vec{k}\cdot n\right)dS \quad . \tag{2.2}$$

Since $\partial\Omega_1(0) \cap \partial\mathcal{D} \supset \partial\Omega_2(0) \cap \partial\mathcal{D} \supset \text{supp } p_1|_{\partial\mathcal{D}}$, $p_1 \geq p_2 \geq 0$ $\forall x \in \mathcal{D}$, and $\frac{\partial}{\partial n}(p_1 - p_2) = 0$ or $p_1 - p_2 = 0$ on $\partial\Omega_2(0) \cap \partial\mathcal{D}$, it follows from (1.31i) and (1.31j) that

$$\frac{d}{dt}|\Omega_1(t) - \Omega_2(t)|\Big|_{t=0} = \int_{\partial\Omega_2(0)\cap\partial\mathcal{D}} \frac{\partial(p_1 - p_2)}{\partial n}dS$$

$$+ \int_{(\partial\Omega_1(0)-\partial\Omega_2(0))\cap\partial\mathcal{D}_1}\left(\frac{\partial p_1}{\partial n} + \vec{k}\cdot n\, dS\right) \leq 0 \quad . \tag{2.3}$$

(2.3) may be combined with lemma 2.1 to prove (2.1).

Remark 2.1: It seems likely that, if we restrict ourselves to components of Ω_1 and Ω_2 which are connected to $\partial\mathcal{D}_0 \cup \partial\mathcal{D}_R$, we can get a stronger relation than (2.3), and prove decay of $|\Omega_1(t) - \Omega_2(t)|$ to 0 as $t \to \infty$. Indeed, in these circumstances it appears that for some constant $b > 0$ we can show that

$$\frac{d}{dt}|\Omega_1(t) - \Omega_2(t)| \leq - b|\Omega_1(t) - \Omega_2(t)| \quad ,$$

in which case

$$|\Omega_1(t) - \Omega_2(t)| \leq e^{-bt}|\Omega_1(0) - \Omega_2(0)| \quad . \qquad (2.4)$$

I have been informed, in the course of writing this up, that a counterexample has been given by Alt for components of Ω_1 and Ω_2 which are not connected to $\partial D_0 \cup \partial D_R$ (Ref. 1).

An immediate consequence of lemma 2.2 is the following: If the prescribed boundary data are time-independent, the initial rate of change of $\Omega(t)$ is the largest, that is,

$$|(\Omega(t + \delta) - \Omega(t)) \cup (\Omega(t) - \Omega(t + \delta))| \leq |(\Omega(\delta) - \Omega(0)) \cup (\Omega(0) - \Omega(\delta))| \quad . \qquad (2.5)$$

Returning to direct consideration of the problem (1.31), we recall our remark regarding its similarity to the anodic smoothing problem. We found (Ref. 8) that the solution of the latter problem, at a time t, could be found directly by finding the steady state of problem (1.7) with u_0 now dependent on the parameter t. Making some alterations and specifications in the steady state version of (1.7) in order that the result will conform to the sort of boundary value problem we have been considering here, we note that the solution to the "anodic smoothing" problem at time t can be found from the solution of the following elliptic free boundary problem:

$$\Delta\xi = 0 \quad , \quad x \in \Omega(0) \subset D \quad , \quad 0 < t < \infty \quad , \tag{2.6a}$$

$$\Delta\xi = 1 \quad , \quad x \in \Omega(t) - \Omega(0) \subset D \quad , \quad 0 < t < \infty \quad , \tag{2.6b}$$

$$\xi \geq 0 \quad , \quad x \in D \quad , \quad 0 < t < \infty \quad , \tag{2.6c}$$

$$\xi = 0 \quad , \quad x \in \partial D_0 \quad , \quad 0 < t < \infty \quad , \tag{2.6d}$$

$$\xi = tp_R > 0 \quad , \quad x \in \partial D_R \quad , \quad 0 < t < \infty \quad , \tag{2.6e}$$

$$\frac{\partial\xi}{\partial n} = t\frac{\partial p}{\partial n} \quad , \quad x \in \partial D_I, \quad 0 < t < \infty \quad , \tag{2.6f}$$

$$\xi \equiv 0 \quad , \quad x \in D - \Omega(t) \quad , \tag{2.6g}$$

$$\xi \text{ and } \nabla\xi \text{ are continuous in } D \quad . \tag{2.6h}$$

Differentiation of each of the conditions (2.6) with respect to t shows that ξ_t satisfies the problem (1.31), except for the gravitational term $-\vec{k}$ in (1.31g). If we transform coordinates and view the problem from a system which is moving downward with unit velocity, the term $-\vec{k}$ will disappear, but then D will appear to be moving upward with unit velocity. Thus, the problem is that of anodic smoothing when the "cathode" is in uniform motion.

It should be noted that the term $-\vec{k}$ in (1.31g) effects a uniform translation of the free surface and has no effect on the character of $\mathrm{Int}(\partial\Omega_f)$ with regard to its differential geometric properties. Thus, the observed relation of the solution of the time-dependent dam problem to the solution of a sequence of steady state Stefan problems in a translating region may be useful primarily for what can then be deduced about the regularity of the free surface from known results for the steady state Stefan problem, and not for practical calculation.

Nevertheless, we can exploit this relation to suggest a numerical approach to the time-dependent dam problem: Viewed from a system translating uniformly downward with unit speed, each point $P \in R^N$ fixed in space is moving upward uniformly with unit speed, and moves through the distance τ in the time interval $[n\tau,(n+1)\tau]$. Let us denote by $\hat{\sigma}^n(\Gamma)$ the points thus swept out by a set $\Gamma \subset R^N$. We define

$$\hat{D}^n \equiv \hat{\sigma}^n(D) \quad , \tag{2.7}$$

a region which, for each n, is fixed with respect to the moving system. Given $\Omega(0) \subset D$, we construct Ω^n according to the following prescription. First we define

$$\hat{\Omega}^0 \equiv \hat{\sigma}^0(\Omega(0)) \subset \hat{D}^0 \quad . \tag{2.8}$$

Then, given $\hat{\Omega}^n$, $n \geq 0$, we find $\overline{\Omega}^{n+1}$ by solving

$$\Delta\xi = 0 \quad , \quad x \in \hat{\Omega}^n \subset \hat{D}^n \quad , \tag{2.9a}$$

$$\Delta\xi = 1 \quad , \quad x \in \overline{\Omega}^{n+1} - \hat{\Omega}^n \subset \hat{D}^n \quad , \tag{2.9b}$$

$$\xi \geq 0 \quad , \quad x \subset \hat{D}^n \quad , \tag{2.9c}$$

$$\xi = 0 \quad , \quad x \in \hat{\sigma}^n(\partial D_0) \cap \partial\hat{D}^n \quad , \tag{2.9d}$$

$$\xi = \tau p_R > 0 \quad , \quad x \in \hat{\sigma}^n(\partial D_R) \cap \partial\hat{D}^n \quad , \tag{2.9e}$$

$$\frac{\partial\xi}{\partial n} = \tau\frac{\partial p}{\partial n} \quad , \quad x \in \hat{\sigma}^n(\partial D_I) \cap \partial\hat{D}^n \quad , \tag{2.9f}$$

$$\xi \equiv 0 \quad , \quad x \in \hat{D}^n - \overline{\Omega}^{n+1} \quad , \tag{2.9g}$$

$$\xi \text{ and } \nabla\xi \text{ are continuous in } \hat{D}^n \quad . \tag{2.9h}$$

$\hat{\Omega}^{n+1}$ is determined from

$$\hat{\Omega}^{n+1} \equiv \overline{\Omega}^{n+1} \cap \hat{D}^{n+1} \quad . \tag{2.10}$$

Finally, the set of points $\hat{\Omega}^n \subset \hat{D}^n$, when viewed from the fixed system at time $n\tau$, is a set Ω^{*n}, and we define

$$\Omega^n \equiv \Omega^{*n} \cap D \quad . \tag{2.11}$$

Viewed from the system at rest, the region $\hat{D}^n$ is seen, in the time interval $[n\tau,(n + 1)\tau)$, as a time-dependent region $D^*(t)$ independent of n and given as follows:

$$D^*(t) = \sigma^*(t)(D) \tag{2.12a}$$

where for any set $\Gamma \subset R^N$, $\sigma^*(t)(\Gamma)$ is defined, for $t \in [n\tau,(n + 1)\tau)$, by

$$\sigma^*(t)(\Gamma) \equiv \{(x,z) | (x,z') \in \Gamma \text{ for some } z' \in [z + t - (n + 1)\tau, z + t - n\tau]\} \quad . \tag{2.12b}$$

Ω^{*n} will then be the region $\Omega(n\tau)$ solving the problem (1.31), with the following modifications:

$$\Omega(0) \text{ is replaced by } \sigma^*(0)\Omega(0) \quad , \tag{2.13a}$$

$$D \text{ is replaced by } D^*(t) \quad , \tag{2.13b}$$

$$\partial D_0 \text{ is replaced by } \sigma^*(t)(\partial D_0) \cap \partial D^*(t) \quad , \tag{2.13c}$$

$$\partial D_R \text{ is replaced by } \sigma^*(t)(\partial D_R) \cap \partial D^*(t) \quad , \tag{2.13d}$$

$$\partial D_I \text{ is replaced by } \sigma^*(t)(\partial D_I) \cap \partial D^*(t) \quad . \tag{2.13e}$$

If ∂D is at all smooth (having, say, piecewise bounded curvature), the same sort of smoothness (up to boundedness of curvature) will characterize $\partial D^*(t)$. Accordingly, *a priori* regularity results will be derivable for p satisfying the modified version of (1.31), and since $D^*(t) \supset D \; \forall t$, one may deduce the deviation between the actual values attained by p and $\frac{\partial p}{\partial n}$ on ∂D for the solution of the modified problem, and the values required in the original problem (1.31). To obtain error estimates for the solution to the modified problem, as opposed to the solution

of (1.31), one need only determine a domain Ω^- known *a priori* to be contained for all time in the union of the supports of p for the modified and original problems, a determination easily made with the help of lemma 2.1, then find the capacities relative to Ω^- of various subsets of $\partial D_0 \cup \partial D_R$ and multiply them by the corresponding errors in the values of p found for those subsets, integrate the error in $\frac{\partial p}{\partial n}$ over ∂D_I, and finally make use of lemma 2.2. For cases of interest, we anticipate an error, as measured by the volume contained between the two determinations of the free boundary, proportional to τ for finite t. If the expectation of remark 2.1 is borne out, such an error estimate will, in fact, hold uniformly for all time.

We recall that, if $\tilde{p}$ satisfies a problem like (1.31) for p, except for the absence of the term $-\vec{k}$ in (1.31g), then

$$\xi(x,z,t) = \int_0^t \tilde{p}(x,z,t')dt' \qquad (2.14)$$

satisfies the elliptic free boundary problem (2.6). This is the transformation used to solve the anodic smoothing problem directly. (It is somewhat reminiscent of the transformation (1.5) relating variants of the Stefan problem.) Since the term $-\vec{k}$ in (1.31g) has been seen to vanish in a coordinate system moving downward with unit velocity, the natural adaptation of (2.14) to the dam problem is the transformation

$$u(x,z,t) = \int_0^t p(x,z+t-t',t')dt' \quad . \qquad (2.15)$$

When a steady state p(x,z) has been achieved as $t \to \infty$, (2.15) becomes

$$u(x,z) = \int_0^\infty p(x,z + t')dt' \quad , \qquad (2.16)$$

which is precisely the Baiocchi transformation (Ref. 2).

Note added in proof: I have learned since doing this work that (2.15) is exactly the same as the transformation introduced by Torelli in a paper by Friedman and Torelli (Ref. 7).

The formal transformation (2.15) is not defined when $(x,z + t - t') \notin \mathcal{D}$ for some $t' \in [0,t]$. (The extension of p to $\mathcal{D} - \Omega(t)$ through the convention that $p \equiv 0$ there is quite natural.) Although a version of (2.15) can be given which is satisfactory for the time-dependent case in a fairly general region $\mathcal{D}$, we shall not consider it here. In the next section, for the steady state case, a version of (2.16) appropriate for more general regions $\mathcal{D}$ will be considered. For the present, let us restrict ourselves to the case in which $\mathcal{D}$ is the cylinder (cf. (1.14))

$$\mathcal{D} = B_0 x (0,\infty) \quad , \qquad (2.17)$$

and also

$$\partial\mathcal{D}_I = B_0 \quad , \qquad (2.18a)$$

$$\partial\mathcal{D}_0 \cup \partial\mathcal{D}_R = \partial B_0 x (0,\infty) \quad . \qquad (2.18b)$$

In this case $(x,z + t - t') \in \mathcal{D} \, \forall t' \in [0,t]$. The time-dependent dam problem (1.31) is then equivalent to the following problem for u:

$$u \text{ and } \nabla u \text{ are continuous in } \mathcal{D} \quad , \qquad (2.19a)$$

$$u \equiv 0 \quad , \quad x \in \mathcal{D} - \Omega(t) \quad , \quad 0 < t < \infty \quad , \qquad (2.19b)$$

$$u \geq 0 \quad , \quad x \in D \quad , \quad 0 < t < \infty \quad , \tag{2.19c}$$

$$u \text{ given} \quad , \quad x \in \partial B_0 x(0,\infty) \quad , \quad 0 < t < \infty \quad , \tag{2.19d}$$

$$u_{tz} - u_{zz} \text{ given} \quad , \quad x \in B_0 \quad , \quad z = 0 \quad , \quad 0 < t < \infty \quad , \tag{2.19e}$$

$$\Delta u(x,z,t) = \begin{cases} 1 \text{ if } (x,z) \in \Omega(t) \text{ and } (x,z+t) \notin \Omega(0) \\ 0 \text{ otherwise} \quad . \end{cases} \tag{2.19f}$$

When a steady state $u(x,z)$ is reached, the problem (2.19) reverts to a form which can be expressed as a variational inequality and which has been considered already in some detail (Ref. 2). This problem is very close to the steady state version of the free boundary problem (1.7)-(1.8), which can be solved by either algorithm I or algorithm II.

3. Steady State

Our first order of business is to give a meaningful version of the transformation (2.16) for a fairly general region D. To this end, we will find it convenient to work with functions satisfying homogeneous conditions on ∂D, and we suppose that we can write

$$p = -\theta_z + \psi \tag{3.1}$$

where θ_z has the following properties:

$$\text{supp}(\theta_z) - C\ell(\partial D_R) - C\ell(\partial D_{I\,2}) \subset \Omega^- \quad , \quad \text{where we know } a\ priori \text{ that } \Omega^- \subset \text{Int}(\Omega) \quad , \tag{3.2a}$$

$$\Delta\theta_z \leq 0 \ \forall x \in D \quad , \tag{3.2b}$$

$$\theta_z = 0 \quad , \quad x \in \partial D_0 \quad , \tag{3.2c}$$

$$\theta_z = - p_R < 0 \quad , \quad x \in \partial D_R \quad , \tag{3.2d}$$

$$\frac{\partial}{\partial n}\theta_z = - \frac{\partial p}{\partial n} \quad , \quad x \in \partial D_{I2} \quad , \tag{3.2e}$$

$$\frac{\partial}{\partial n}\theta_z = 0 \quad , \quad x \in \partial D_{I1} \quad . \tag{3.2f}$$

Obviously, from (1.31) and (3.2)

$$\psi = 0 \quad , \quad x \in \partial D_0 \cup \partial D_R \quad , \tag{3.3a}$$

$$\frac{\partial \psi}{\partial n} = 0 \quad , \quad x \in \partial D_{I2} \quad , \tag{3.3b}$$

$$\Delta\psi = \Delta\theta_z \leq 0 \quad , \quad x \in \Omega \quad , \tag{3.3c}$$

$$\psi = 0 \quad , \quad x \in \partial\Omega_f \quad , \tag{3.3d}$$

$$\frac{\partial \psi}{\partial n} = - \vec{k} \cdot n \quad , \quad x \in \partial D_{I1} \cap \partial\Omega \quad . \tag{3.3e}$$

As noted after equation (1.13g), we have $\psi \geq 0$ on $\partial D_{I1} \cap \partial\Omega$. In addition, since $p \geq 0$ in D and $p = 0$ on $\partial\Omega_f$, we get from (3.2) and (1.31g) that

$$- \frac{\partial \psi}{\partial n} = \vec{k} \cdot n \geq 0 \quad , \quad x \in \partial\Omega_f \quad . \tag{3.3f}$$

All these conditions imply that

$$\psi \geq 0 \quad , \quad x \in \Omega \quad . \tag{3.3g}$$

It is now natural for us to make the extension

$$\psi \equiv 0 \quad , \quad x \in D - \Omega \quad . \tag{3.4}$$

Defining, for all $x \in R^{N-1}$,

$$Z(x) \equiv \sup\{z \,|\, (x,z) \in D\} \quad , \tag{3.5}$$

we are led, in our version of the Baiocchi transformation, to consider the function

$$\zeta(x,z) = \int_z^{Z(x)} \psi(x,z')dz' \quad . \tag{3.6}$$

We will rewrite the steady state problem as a problem for ζ, in a form for which a solution algorithm readily suggests itself. Before we proceed, however, it will be useful to distinguish several cases which may arise. It will prove helpful for us to make the definitions, for $x \in R^{N-1}$,

$$\{z_\Omega(x)\} = \partial\{z|(x,z) \in \Omega\} \quad , \tag{3.7a}$$

$$z^*(x) = \sup(z \in \{z_\Omega(x)\}) \quad , \tag{3.7b}$$

$$\{z_f(x)\} = \{z|(x,z) \in \partial\Omega_f\} \quad . \tag{3.8}$$

Also we define the set

$$\{z_D(x)\} = \partial\{z|(x,z) \in D\} \quad , \tag{3.9a}$$

whose members are written as

$$z_1(x) > z_2(x) > \ldots > z_{n(x)}(x) \quad . \tag{3.9b}$$

If $Z(x) = \infty$, $n(x)$ is odd, and on account of (3.3e) elements of $\{z_f(x)\}$ can only lie between ∞ and $z_1(x)$, or between $z_{2i}(x)$ and $z_{2i+1}(x)$ for

$$i \in S_D(x) \equiv \left\{ i|1 \le i \le \frac{n(x)-1}{2} \quad , \quad (x,z_{2i}(x)) \in \partial D_1 \cup \partial D_{I1} \right\} \quad . \tag{3.10a}$$

If $Z(x) < \infty$, $n(x)$ is even, and elements of $\{z_f(x)\}$ can only lie between $z_{2i-1}(x)$ and $z_{2i}(x)$ for

$$i \in S_D(x) \equiv \left\{ i|1 \le i \le \frac{n(x)}{2} \quad , \quad (x,z_{2i-1}(x)) \in \partial D_1 \cup \partial D_{I1} \right\} \quad . \tag{3.10b}$$

If $n(x)$ is 0, 1, or 2, we see that the number of elements of $\{z_\Omega(x)\}$ is 0 or 2. In the former case $\{z_f(x)\}$ is empty; in the

latter it is empty or contains at most one element, $z^*(x)$. Note that if D is convex, we have that $n(x)$ is 0, 1, or 2 for all x.

We shall now restrict ourselves to the case where for each x the set $\{z_f(x)\}$ contains at most one element, $z^*(x)$, and also where $\forall x$

$$\{(x,z) \in C\ell(D) \mid z = Z(x)\} \subset \partial D_1 \cup \partial D_R \cup \partial D_I \quad . \tag{3.11}$$

Since the more general situation may be posed as a mathematical problem, we will say a few words about it at the end of this section. However, there do not seem to be any interesting or important problems of flows through dams that are excluded by the restriction we make here.

From (3.11) we have

$$T \equiv \{(x,z) \in C\ell(\Omega) \mid z = z^*(x)\} \subset \partial D_1 \cup \partial D_R \cup \partial D_I \cup \partial\Omega_f \quad , \tag{3.12}$$

and thus by using (1.31j) and (3.3) we conclude that at the point $(x,z^*(x))$, either

$$-\frac{\partial\psi}{\partial n} \geq \vec{k} \cdot n \quad , \quad -\frac{\partial\psi}{\partial n} = \vec{k} \cdot n \quad , \quad \text{or } \psi > 0 \quad . \tag{3.13}$$

The fact that $\psi > 0$ on ∂D_{I2} for sufficiently smooth ∂D (for example, with bounded curvature) follows by getting a contradiction from the assumption that $\psi = 0$ at any point of this set.

Using

$$\zeta(x,z) = \int_z^{z^*(x)} \psi(x,z')dz' \tag{3.14}$$

and (3.3), and setting $\zeta(x,z) \equiv 0$ for $z > z^*(x)$, we get

$$\Delta\zeta(x,z) = -\Delta\theta(x,z) + \Delta\theta(x,z^*(x)) - \frac{1}{n\cdot\vec{k}}\frac{\partial\psi}{\partial n}(x,z^*(x))$$

$$+ \nabla\cdot(\psi(x,z^*(x))\nabla z^*(x)) + \frac{\psi(x,z^*(x))}{n\cdot\vec{k}}\delta_{\partial D} \quad (3.15)$$

in the region $D \cup \{(x,z)|z \geq z^*(x)\}$. Here $\delta_{\partial D}$ is a Dirac measure on ∂D.

On $\partial\Omega_f \cup (T \cap \mathrm{Int}(\partial D_1))$

$$\zeta = \nabla\zeta = 0 \quad , \quad (3.16a)$$

and just below this set

$$\Delta\zeta = -\frac{1}{n\cdot\vec{k}}\frac{\partial\psi}{\partial n}(x,z^*(x)) \geq 1 \quad , \quad (3.16b)$$

where we get equality just below $\partial\Omega_f$. Just below $T \cap \partial D_{I1}$, (3.15) becomes

$$\Delta\zeta = 1 + \nabla\cdot(\psi(x,z^*(x))\nabla z^*(x)) + \frac{\psi(x,z^*(x))}{n\cdot\vec{k}}\delta_{\partial D} \quad ,(3.16c)$$

and just below $T \cap \partial D_{I2}$,

$$\psi > 0 \quad , \quad (3.16d)$$

$$\Delta\zeta = \nabla\cdot(\psi(x,z^*(x))\nabla z^*(x)) + \frac{\psi(x,z^*(x))}{n\cdot\vec{k}}\delta_{\partial D} \quad . \quad (3.16e)$$

The problem for ζ near $\partial\Omega_f$, as given by (3.16a) and (3.16b), is close to the steady state free boundary problem described in (1.7) and (1.8). It has been seen (Ref. 3) that this free boundary problem may be viewed as the limit as $\epsilon \to 0$ of a nonlinear problem dependent on the positive parameter ϵ and defined on the whole region D. Since (3.16b) shows that $\Delta\zeta$ is not necessarily constant near $\partial\Omega_f \cup (T \cap \mathrm{Int}(\partial D_1))$, the treatment previously

given is modified slightly. Near $\partial\Omega_f \cup (T \cap \mathrm{Int}(\partial\mathcal{D}_1))$ we are led to consider the ϵ-dependent problem

$$\Delta\zeta(\epsilon) + \min\left(\frac{1}{n \cdot \vec{k}}\frac{\partial}{\partial n}\psi(x,z^*(x)),-1\right)g_\epsilon(\zeta(\epsilon)) = 0 \quad ,(3.17)$$

where

$$g_\epsilon(\zeta) \equiv \begin{cases} 1 & \zeta \geq \epsilon \\ \frac{\zeta}{\epsilon} & 0 \leq \zeta \leq \epsilon \end{cases} \quad . \qquad (3.18)$$

The boundary condition is that $\zeta(\epsilon) \to 0$ as $(x,z) \to (x,Z(x))$. We may consider the limit as $\epsilon \to 0$ of $\zeta(\epsilon)$ in (3.17) to be the solution of the equation

$$\Delta\zeta + \begin{cases} \min\left(\frac{1}{n \cdot \vec{k}}\frac{\partial}{\partial n}\psi(x,z^*(x)),-1\right) & \zeta > 0 \\ 0 & \zeta = 0 \end{cases} = 0 \quad .(3.19)$$

By using monotonicity arguments for elliptic equations we can show that, if ζ and $\zeta(\epsilon)$ satisfy the same elliptic boundary value problem in a region, with the same boundary data, except for the replacement of (3.16b) by (3.17) in parts of the region, then

$$\zeta \leq \zeta(\epsilon) \leq \zeta + \epsilon \quad . \qquad (3.20)$$

We now want to look at the behavior of $\zeta(\epsilon)$ where $0 \leq \zeta(\epsilon) \leq \epsilon$, and for x such that $(x,z^*(x)) \in \partial\Omega_f \cup (T \cap \mathrm{Int}(\partial\mathcal{D}_1))$. To facilitate the discussion, let us make the definition, for a function u, of

$$S_\epsilon(u) \equiv \{(x,z) \in \mathcal{D} | 0 \leq u \leq \epsilon\} \quad . \qquad (3.21)$$

It follows from (3.20) that $S_\epsilon(\zeta(\epsilon)) \subset S_\epsilon(\zeta)$. In addition we define

$$A \equiv \{(x,z) \in D \,|\, (x,z^*(x)) \in \partial\Omega_f \cup (T \cap \mathrm{Int}(\partial D_1))\} \quad . \tag{3.22}$$

If $\left|\frac{1}{n \cdot \vec{k}} \frac{\partial\psi}{\partial n}(x,z^*(x))\right|$ is bounded for $x \in T \cap \mathrm{Int}(\partial D_1)$, one sees that $|\Delta\zeta(\epsilon)|$ is bounded, *uniformly in* ϵ, in $A \cap S_\epsilon(\zeta)$, and thus one gets equicontinuity of the derivatives $\nabla\zeta(\epsilon)$. Hence

$$\nabla\zeta(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} \nabla\zeta \text{ pointwise} \quad , \quad (x,z) \in S_\epsilon(\zeta) \cap A \quad . \tag{3.23}$$

Since

$$\sup_{A \cap S_\epsilon(\zeta)} \psi = \sup_{A \cap S_\epsilon(\zeta)} (-\zeta_z) \to 0 \text{ as } \epsilon \to 0 \quad , \tag{3.24}$$

(3.23) immediately implies that

$$\psi(\epsilon) \to 0 \text{ as } \epsilon \to 0 \quad , \quad (x,z) \in S_\epsilon(\zeta(\epsilon)) \cap A \quad . \tag{3.25}$$

Let the equation of the surface $\zeta(\epsilon) = \epsilon$ be written in the form $z = z_\epsilon(x)$, that is,

$$\zeta(\epsilon,x,z_\epsilon(x)) = \epsilon \quad . \tag{3.26}$$

Then on the intersection of this surface with A, it will follow from (3.25) and (3.17) that we have in the limit as $\epsilon \to 0$

$$-\frac{1}{n \cdot \vec{k}} \frac{\partial}{\partial n}\psi(\epsilon,x,z_\epsilon(x)) + \Delta\zeta(\epsilon,x,z_\epsilon(x)) =$$

$$- \min\left(\frac{1}{n \cdot \vec{k}} \frac{\partial\psi(x,z^*(x))}{\partial n}, -1\right) \quad . \tag{3.27}$$

Similarly, we get from (3.25) and (3.17)

$$-\frac{1}{n \cdot \vec{k}} \frac{\partial}{\partial n}\psi(\epsilon,x,z) \underset{\epsilon\to 0}{\longrightarrow} \Delta\zeta(\epsilon,x,z) \xrightarrow[(x,z)\to(x,Z(x))]{} 0 \quad . \tag{3.28}$$

Corresponding to (3.17) for $\zeta(\epsilon)$ we find the following equation for $\psi(\epsilon)$ near $\partial\Omega_f \cup (T \cap \mathrm{Int}(\partial D_1))$:

$$\Delta\psi(\epsilon) + \min\left(\frac{1}{n \cdot \vec{k}} \frac{\partial}{\partial n}\psi(x,z^*(x)),-1\right) \begin{cases} \dfrac{\psi(\epsilon)}{\epsilon} & 0 \leq \zeta(\epsilon) < \epsilon \\ 0 & \zeta(\epsilon) > \epsilon \end{cases} = 0 \quad . \tag{3.29}$$

This equation is not useful to compute with, except for certain special types of boundary data and regions D, such as those occurring in the simplest of the problems studied by Baiocchi and Magenes (Ref. 2), because of the appearance, at points on $\partial D \cap \partial\Omega$, of the term $\frac{\partial}{\partial n}\psi(x,z^*(x))$, which is, after all, an unknown of the problem.

On the other hand, difficulties associated with the fact that ∂D and $\partial\Omega$ have a nonempty intersection should be no difficulties at all, because in that regard the free-boundary aspect of the problem disappears, and the solution of the problem is straightforward when the region occupied by the fluid is known. Thus, in order to see what freedom we may have in using versions of the equation which differ from (3.29) and may be more amenable to numerical implementation, let us examine the case when $\partial\Omega_f = \emptyset$.

Suppose we want to solve

$$\Delta\psi = 0 \quad , \quad x \in D \quad , \tag{3.30a}$$

with

$$\psi(x,z) \to 0 \text{ for } (x,z) \to (x,Z(x)) \quad , \quad x \in \Gamma \quad , \tag{3.30b}$$

and other data imposed to make $\psi \geq 0$ in D. This is not a free boundary problem, so that (3.3f) cannot be imposed. For ζ given

by (3.6),

$$\Delta\zeta = -\frac{1}{n \cdot \vec{k}} \frac{\partial\psi}{\partial n}(x,Z(x)) \quad , \quad x \in \Gamma \quad . \qquad (3.31)$$

This can also be viewed as the limit as $\epsilon \to 0$ of the solution of

$$\Delta\zeta(\epsilon) + \frac{1}{n \cdot \vec{k}} \frac{\partial\psi}{\partial n}(x,Z(x))g_\epsilon(\zeta(\epsilon)) = 0 \quad , \quad x \in \Gamma \quad , \qquad (3.32a)$$

$$\zeta(\epsilon,x,z) \to 0 \text{ for } (x,z) \to (x,Z(x)) \quad , \quad x \in \Gamma \quad , \qquad (3.32b)$$

with $\zeta(\epsilon) \geq 0$ in D. As before, we find that (3.25) holds for all points for which $x \in \Gamma$ and $\zeta(\epsilon) \leq \epsilon$. Hence as $\epsilon \to 0$ we expect

$$-\frac{1}{n \cdot \vec{k}} \frac{\partial\psi(\epsilon,x,z_\epsilon(x))}{\partial n} \to \Delta\zeta(\epsilon,x,z_\epsilon(x)) = -\frac{1}{n \cdot \vec{k}} \frac{\partial\psi}{\partial n}(x,Z(x)) \quad . \qquad (3.33)$$

In addition, it follows from (3.25) and (3.32a) that

$$-\frac{1}{n \cdot \vec{k}} \frac{\partial\psi(\epsilon,x,Z(x))}{\partial n} \to \Delta\zeta(\epsilon,x,Z(x)) = 0 \quad . \qquad (3.34)$$

Differentiation of (3.32a) with respect to z yields

$$\Delta\psi(\epsilon) + \frac{1}{n \cdot \vec{k}} \frac{\partial\psi}{\partial n}(x,Z(x)) \begin{cases} \dfrac{\psi(\epsilon)}{\epsilon} & 0 \leq \zeta(\epsilon) < \epsilon \\ 0 & \zeta(\epsilon) > \epsilon \end{cases} = 0 \quad , \quad x \in \Gamma \quad . \qquad (3.35)$$

Consider now the following variation on (3.32):

$$\Delta\zeta(\epsilon) - h(x)g_\epsilon(\zeta(\epsilon)) + \left(\frac{1}{n \cdot \vec{k}} \frac{\partial\psi}{\partial n}(x,Z(x)) + h(x)\right) = 0 \quad , \quad x \in \Gamma \quad , \qquad (3.36a)$$

$$\zeta(\epsilon,x,z) \to 0 \text{ for } (x,z) \to (x,Z(x)) \quad , \quad x \in \Gamma \quad , \qquad (3.36b)$$

with $\zeta(\epsilon) \geq 0$ in D. $h(x)$ is constrained by the inequalities

$$0 < h(x) \leq -\frac{1}{n \cdot \vec{k}} \frac{\partial\psi}{\partial n}(x,Z(x)) \quad , \quad x \in \Gamma \quad . \qquad (3.37)$$

The same argument that led to (3.25) holds again, for $x \in \Gamma$ and $\zeta(\epsilon) \leq \epsilon$. Thus we get as $\epsilon \to 0$

$$-\frac{1}{n \cdot \vec{k}} \frac{\partial}{\partial n} \psi(\epsilon, x, z_\epsilon(x)) + \Delta\zeta(\epsilon, x, z_\epsilon(x)) =$$

$$-\frac{1}{n \cdot \vec{k}} \frac{\partial \psi}{\partial n}(x, Z(x)) \quad , \quad x \in \Gamma \quad , \qquad (3.38a)$$

and

$$-\frac{1}{n \cdot \vec{k}} \frac{\partial}{\partial n} \psi(\epsilon, x, Z(x)) + \Delta\zeta(\epsilon, x, Z(x)) =$$

$$-\frac{1}{n \cdot \vec{k}} \frac{\partial \psi}{\partial n}(x, Z(x)) - h(x) \quad , \quad x \in \Gamma \quad . \qquad (3.38b)$$

Differentation of (3.36a) gives us the following equation for $\psi(\epsilon)$:

$$\Delta\psi(\epsilon) - h(x) \begin{cases} \dfrac{\psi(\epsilon)}{\epsilon} & 0 \leq \zeta(\epsilon) < \epsilon \\ 0 & \zeta(\epsilon) > \epsilon \end{cases} = 0 \quad , \quad x \in \Gamma \quad . \qquad (3.39)$$

Comparing (3.33) and (3.38a), we observe that we get the same result in the limit whether we use (3.35) or (3.39), as long as $h(x)$ satisfies the constraint (3.37).

Now let us return to our problem. We see that we can compute $\psi(\epsilon)$, near $\partial\Omega_f \cup (T \cap \mathrm{Int}(\partial\mathcal{D}_1))$, from the equation

$$\Delta\psi(\epsilon) - \begin{cases} \dfrac{\psi(\epsilon)}{\epsilon} & 0 \leq \zeta(\epsilon) < \epsilon \\ 0 & \zeta(\epsilon) > \epsilon \end{cases} = 0 \quad . \qquad (3.40)$$

For the more general situation described by (3.15), we are then led to consider

$$\Delta\psi(\epsilon) - \Delta\theta_z(x,z) - \begin{cases} \dfrac{\psi(\epsilon)}{\epsilon} & 0 \leq \zeta(\epsilon) < \epsilon \\ 0 & \zeta(\epsilon) > \epsilon \end{cases} = 0 \quad , \quad (x,z) \in D \quad . \tag{3.41}$$

Corresponding to (3.41) we have the equation for $\zeta(\epsilon)$,

$$\Delta\zeta(\epsilon) + \Delta\theta(x,z) - \Delta\theta(x,Z(x)) - g_\epsilon(\zeta(\epsilon)) + \frac{1}{n \cdot \vec{k}} \frac{\partial}{\partial n}\psi(\epsilon,x,Z(x))$$

$$- \nabla \cdot (\psi(\epsilon,x,Z(x))\nabla Z(x)) - \frac{1}{n \cdot \vec{k}}\psi(\epsilon,x,Z(x))\delta_{\partial D} = 0 \tag{3.42}$$

for $(x,z) \in D \cup \{(x,z)|z \geq Z(x)\}$. Here, of course, $\zeta(\epsilon)$ is given by

$$\zeta(\epsilon,x,z) = \int_z^{Z(x)} \psi(\epsilon,x,z')dz' \quad . \tag{3.43}$$

In (3.42), at values of x for which $z^*(x) = Z(x)$ and we have prescribed boundary conditions on $\psi(x,z^*(x))$, the terms $\frac{1}{n \cdot \vec{k}} \frac{\partial}{\partial n}\psi(\epsilon,x,Z(x)) - \nabla \cdot (\psi(\epsilon,x,Z(x))\nabla Z(x)) - \frac{1}{n \cdot \vec{k}}\psi(\epsilon,x,Z(x))\delta_{\partial D}$ will adjust themselves in the limit as $\epsilon \to 0$ to leave the equation, when expressed at $z = z_\epsilon(x)$, invariant under a suitable change in the coefficient of g_ϵ. However, for values of x such that $Z(x) \neq z^*(x)$, that is, such that $(x,z^*(x)) \in \partial\Omega_f$, such an adjustment cannot take place, because in the limit as $\epsilon \to 0$ we will invariably have

$$\psi(\epsilon,x,Z(x)) \to 0 \quad , \quad \frac{1}{n \cdot \vec{k}} \frac{\partial}{\partial n}\psi(\epsilon,x,Z(x)) \to 0 \quad . \tag{3.44}$$

Thus, the coefficient of g_ϵ cannot be changed for such values of x without changing the problem in the limit $\epsilon \to 0$. In rough

physical terms, this situation arises because on $T \cap \partial D$ we have an outflow, under assumption (3.11), and we can do whatever we want with the flow as it leaves the region, as long as we do not create disturbances which propagate back into the region. (When (3.11) is not satisfied and we have a point $\in \partial D_2$ at which $z^*(x) = Z(x)$ with an inflow, we are still free to make suitable adjustments in the coefficient of g_ϵ in (3.42), as the inflow is effectively prescribed at such points, through the boundary conditions, independently of such variations.) The limit as $\epsilon \to 0$ of $\zeta(\epsilon)$ will satisfy

$$\Delta\zeta + \Delta\theta(x,z) - \Delta\theta(x,Z(x)) + \frac{1}{n \cdot \vec{k}} \lim_{\epsilon\to 0} \frac{\partial}{\partial n}\psi(\epsilon,x,Z(x))$$

$$- \lim_{\epsilon\to 0} \nabla \cdot (\psi(\epsilon,x,Z(x))\nabla Z(x)) - \frac{1}{n \cdot \vec{k}} \lim_{\epsilon\to 0} \psi(\epsilon,x,Z(x))\delta_{\partial D}$$

$$- \begin{cases} 1 & \zeta > 0 \\ 0 & \zeta = 0 \end{cases} = 0 \quad , \quad (3.45)$$

where $\psi(\epsilon)$ is given by (3.41) and appropriate boundary conditions on ∂D.

For the purpose of solving (3.41) and (3.42) numerically, we introduce a pseudo-time variable which we label "t," but which should be distinguished from the real time appearing earlier in this paper. Thus, we write in place of (3.41) and (3.42)

$$\psi_t(\epsilon,t) = \Delta\psi(\epsilon,t) - \Delta\theta_z(x,z)$$

$$- \begin{cases} \dfrac{\psi(\epsilon,t)}{\epsilon} & 0 \le \zeta(\epsilon,t) < \epsilon \\ 0 & \zeta(\epsilon,t) > \epsilon \end{cases} \quad , \quad (x,z,t) \in D\times(0,\infty) \quad , \quad (3.46)$$

and

$$\zeta_t(\epsilon,t) = \Delta\zeta(\epsilon,t) + \Delta\theta(x,z) - \Delta\theta(x,Z(x)) - g_\epsilon(\zeta(\epsilon,t)$$

$$+ \frac{1}{n \cdot \vec{k}} \frac{\partial}{\partial n}\psi(\epsilon,t,x,Z(x)) - \nabla \cdot (\psi(\epsilon,t,x,Z(x))\nabla Z(x))$$

$$- \frac{1}{n \cdot \vec{k}}\psi(\epsilon,t,x,Z(x))\delta_{\partial D} \quad , \quad (x,z,t) \in D\times(0,\infty) \quad . \tag{3.47}$$

We are now ready to discuss numerical solution of the problem. However, before we proceed, we should note one further complication that arises. This is that in the problem (3.3) for ψ in the region Ω, which we may now replace by a problem for ψ in D through (3.4) and the interpretation of ψ as the limit of $\psi(\epsilon)$ in (3.41), all the boundary data for ψ on ∂D are homogeneous with the exception of the data on $\partial D_{I_1} \cap \partial\Omega$, given by (3.3e). The difficulty is that we do not generally know the extent of this set until we have solved the problem. Hence, we shall assume for the immediate future that $\partial D_{I_1} = \emptyset$, and see how to solve the problem in that case. Then we will show how an elementary modification of the algorithm used for the solution when $\partial D_{I_1} = \emptyset$ will yield the solution for the practically important case $\partial D_{I_1} \neq \emptyset$. (There are, however, cases where we know *a priori* that $\partial D_{I_1} \subset \partial\Omega$. In such situations we can make the appropriate changes in θ, defined in (3.2), to get, for $\psi = p + \theta_z$, $\frac{\partial\psi}{\partial n} = 0$, $x \in \partial D_{I_1}$, instead of (3.3e). Then the algorithm which we give presently will be applicable without any modification. Such a case occurs when $\partial D_{I_1} = B_0$, given by (1.14).)

As a preliminary to the discussion of approximate solutions of the dam problem, we introduce some operators. In view of the boundary conditions imposed on ψ in (3.3) and our tentative assumption that $\partial D_{I_1} = \emptyset$, we start with the operator $S_0(t)$ defined as follows:

$$\xi(t) = S_0(t)\xi_0 \tag{3.48a}$$

is the solution of the initial value problem

$$\xi_t = \Delta\xi \quad , \quad (x,t) \in D\times(0,\infty) \quad , \tag{3.48b}$$

$$\xi = 0 \quad , \quad (x,t) \in (\partial D_0 \cup \partial D_R)\times(0,\infty) \quad , \tag{3.48c}$$

$$\frac{\partial\xi}{\partial n} = 0 \quad , \quad (x,t) \in \partial D_I\times(0,\infty) \quad , \tag{3.48d}$$

$$\xi(t) \xrightarrow[t\downarrow 0]{} \xi_0 \quad , \quad x \in D \quad . \tag{3.48e}$$

It follows, then, that the solution of

$$\psi_t = \Delta\psi - \Delta\theta_z \quad , \quad (x,t) \in D\times(0,\infty) \quad , \tag{3.49a}$$

$$\psi = 0 \quad , \quad (x,t) \in (\partial D_0 \cup \partial D_R)\times(0,\infty) \quad , \tag{3.49b}$$

$$\frac{\partial\psi}{\partial n} = 0 \quad , \quad (x,t) \in \partial D_I\times(0,\infty) \quad , \tag{3.49c}$$

$$\psi(t) \xrightarrow[t\downarrow 0]{} \psi_0 \quad , \tag{3.49d}$$

is given by

$$\psi(t) = S_0(t)\psi_0 - \left(\int_0^t S_0(t')dt'\right)\Delta\theta_z \quad , \tag{3.49e}$$

when $\Delta\theta_z$ is independent of t.

Next, let us introduce the operator S(t) defined by

$$(S(t)\zeta_0)(x,z) \equiv \int_z^{Z(x)} \left\{(S_0(t)(-\zeta_{0z}))(x,z') - \left(\left(\int_0^t S_0(t')dt'\right)\Delta\theta_z\right)(x,z')\right\}dz' \quad . \tag{3.50}$$

$S(t)$ is a semi-group:

$$S(t_1 + t_2) = S(t_1)S(t_2) \quad . \tag{3.51}$$

Remark 3.1: In various problems we may want to work with one of the norms

$$N_1(u) \equiv \int_D |u_z| dxdz \tag{3.52a}$$

or

$$N_2(u) \equiv \sup_{x \in R^{N-1}} \int_0^\infty |u_z(x,z)| dz \quad . \tag{3.52b}$$

We observe that the operator $S(t)$ is contractive in N_1:

$$N_1(S(t)u_1 - S(t)u_2) \le N_1(u_1 - u_2) \quad . \tag{3.53}$$

In general this is not true in N_2. A case for which $S(t)$ is contractive with the norm N_2 is the case $\partial D_I = B_0$, as given in (1.14).

Remark 3.2: Except for special cases, such as that for which

$$\{x|(x,z_2) \in C\ell(D)\} \supset \{x|(x,z_1) \in C\ell(D)\} \text{ if } z_2 \ge z_1 \text{ and } \partial D_I = B_0 \quad , \tag{3.54}$$

it is *not* generally true that the operator $S(t)$ is monotone:

$$\zeta_1(x,Z(x)) = \zeta_2(x,Z(x)) = 0 \quad , \quad \zeta_1 \ge \zeta_2$$
$$\not\Rightarrow (S(t)\zeta_1)(x,Z(x)) = (S(t)\zeta_2)(x,Z(x)) = 0 \quad ,$$
$$S(t)\zeta_1 \ge S(t)\zeta_2 \quad . \tag{3.55}$$

The case (3.54) does not appear to be of practical interest. However, we have a different type of monotonicity:

$$\zeta_1(x,Z(x)) = \zeta_2(x,Z(x)) = 0 \quad , \quad \zeta_1 \ge \zeta_2 \quad , \quad \zeta_{1z} \le \zeta_{2z}$$
$$\Longrightarrow (S(t)\zeta_1)(x,Z(x)) = (S(t)\zeta_2)(x,Z(x)) = 0 \quad ,$$
$$S(t)\zeta_1 \ge S(t)\zeta_2 \quad , \quad (S(t)\zeta_1)_z \le (S(t)\zeta_2)_z \quad . \tag{3.56}$$

We also introduce the operator $P(\tau)(\tau > 0)$:

$$P(\tau)\zeta \equiv \zeta - \tau \qquad (3.57)$$

and the operator M:

$$M\zeta \equiv \max(\zeta,0). \qquad (3.58)$$

(With this notation, (1.9c) can be written

$$v^{n+1} = MP(\tau)S_2(\tau)v^n \quad . \qquad (3.59))$$

Remark 3.3: It is not generally true that the operator $MP(\tau)$ is contractive in either of the norms N_i, $i = 1, 2$, given in (3.52). However, if $\zeta_{1z} \leq 0$, $\zeta_{2z} \leq 0$, and $\zeta_1(x,Z(x)) = \zeta_2(x,Z(x)) = 0$, we have

$$N_i(MP(\tau)\zeta_1 - MP(\tau)\zeta_2) \leq N_i(\zeta_1 - \zeta_2) \quad , \quad i = 1, 2. \qquad (3.60)$$

Also, monotonicity holds:

$$\zeta_1(x,Z(x)) = \zeta_2(x,Z(x)) = 0 \quad , \quad \zeta_1 \geq \zeta_2 \quad , \quad \zeta_{1z} \leq \zeta_{2z}$$
$$\Longrightarrow (MP(\tau)\zeta_1)(x,Z(x)) = (MP(\tau)\zeta_2)(x,Z(x)) = 0 \quad , \quad MP(\tau)\zeta_1 \geq MP(\tau)\zeta_2 \quad , \quad (MP(\tau)\zeta_1)_z \leq (MP(\tau)\zeta_2)_z \quad . \qquad (3.61)$$

Because of the frequency with which we will use the monotonicity relations (3.56) and (3.61), we shall use a special notation:

$$\zeta_1 \overset{*}{\geq} \zeta_2 \text{ if and only if } \zeta_1(x,Z(x)) = \zeta_2(x,Z(x)) = 0 \quad , \quad \zeta_1 \geq \zeta_2 \quad , \text{ and } \zeta_{1z} \leq \zeta_{2z} \quad . \qquad (3.62)$$

In words we will say that ζ_1 is more * than or equal to ζ_2, ζ_2 is less * than or equal to ζ_1, etc.

Let us now turn to the dam problem, taken to be the limit of (3.46), (3.47) as $\epsilon \to 0$. We generate approximate functions $\psi^n(x,z)$ and $\zeta^n(x,z)$, which are supposed to approximate $\psi(x,z,n\tau)$ and $\zeta(x,z,n\tau)$, respectively:

$$\psi^n(x,z) \sim \psi(x,z,n\tau) \quad , \tag{3.63a}$$

$$\zeta^n(x,z) \sim \zeta(x,z,n\tau) \quad . \tag{3.63b}$$

ψ^n and ζ^n will be required to have the following properties:

$$\zeta^n(x,Z(x)) = 0 \quad , \tag{3.64a}$$

$$\psi^n \geq 0 \quad , \tag{3.64b}$$

$$\zeta^n \geq 0 \quad . \tag{3.64c}$$

Our solution of (3.46), (3,47) is done by a "split-step" scheme. Note that if (3.46) were just

$$\psi_t = \Delta\psi - \Delta\theta_z \quad ,$$

with the boundary conditions (3.3) and the assumption $\partial\mathcal{D}_{I1} = \emptyset$, we would have simply

$$\psi((n+1)\tau) = -\left(S(\tau)\left(\int_z^{Z(x)} \psi(x,z',n\tau)dz'\right)\right)_z \quad . \tag{3.65}$$

In that case,

$$\zeta((n+1)\tau) = S(\tau)\zeta(n\tau) \tag{3.66}$$

would solve (3.47) except for the term $-g_\epsilon(\zeta)$. (Note that

$$\zeta \geq^* 0 \Longrightarrow S(t)\zeta \geq^* 0 \tag{3.67}$$

as long as $\Delta\theta_z \leq 0$, as required in (3.2b).) We approximate the effect of the term $-g_\epsilon(\zeta)$, acting over a time τ, on ζ by the operator $P(\tau)$. Then, in view of the fact that we are to have $\zeta^{n+1} \geq^* 0$, we operate on the result by M, since any violations of

this condition can only be due to errors in the split-step scheme. Thus, our algorithm is

$$\zeta^{n+1} = MP(\tau)S(\tau)\zeta^{n} = F^{n+1}\zeta_0 \quad , \tag{3.68a}$$

where

$$F\zeta \equiv MP(\tau)S(\tau)\zeta \quad . \tag{3.68b}$$

Thus the procedure for solving the dam problem is just a variation on algorithm II of section 1 for the diffusion-consumption problem. This method has been called the "truncation method" in the literature (Ref. 3).

It is not hard, given a particular dam D and particular boundary conditions, to find functions ζ^- and ζ^+ such that

$$F\zeta^- \overset{*}{\geq} \zeta^- \quad , \tag{3.69a}$$

$$F\zeta^+ \overset{*}{\leq} \zeta^+ \quad . \tag{3.69b}$$

(For example, $\zeta^- \equiv 0$ always works; so does $\zeta^+(x,z) = \int_z^{\tilde{Z}(x)} \psi^+(x,z')dz'$ where $\psi^+(x,z)$ satisfies

$$\Delta\psi^+ = \Delta\theta_z \quad , \quad x \in D \quad ,$$

$$\psi^+ = 0 \quad , \quad x \in \partial D_0 \cup \partial D_R \quad ,$$

$$\frac{\partial\psi^+}{\partial n} = 0 \quad , \quad x \in \partial D_I \quad ,$$

and $\tilde{Z}(x)$ may be taken to be $Z(x)$ when $Z(x) < \infty$; $\tilde{Z}$ may be taken as a suitable finite bound when $Z(x) = \infty$, chosen so that the integral for ζ^+ is finite.) Then the sequence $\{F^n\zeta^-\}$ will be nondecreasing * and the sequence $\{F^n\zeta^+\}$ will be nonincreasing *. Further, with the definitions

$$\min * (\zeta_1,\zeta_2) \equiv \int_z^{Z(x)} \min(-\zeta_{1z}(x,z') \quad , \quad -\zeta_{2z}(x,z')dz' \quad , \tag{3.70a}$$

$$\max{}^*(\zeta_1,\zeta_2) \equiv \int_z^{Z(x)} \max(-\zeta_{1z}(x,z') \quad , \quad -\zeta_{2z}(x,z'))dz' \quad , \tag{3.70b}$$

we see that

$$F^n \max{}^*(\zeta^-,\zeta^+) \geq^* \max{}^*(F^n\zeta^-,F^n\zeta^+) \quad . \tag{3.71}$$

From the contractiveness of F in N_1, as shown by (3.53) and (3.60), one may deduce

$$\begin{aligned} N_1(F^n\zeta^-) \leq N_1(\max{}^*(F^n\zeta^-,F^n\zeta^+)) &\leq N_1(\max{}^*(\zeta^-,\zeta^+)) \\ &\leq N_1(\zeta^-) + N_1(\zeta^+) \quad , \end{aligned} \tag{3.72}$$

so that the nondecreasing * sequence $\{F^n\zeta^-\}$ is bounded from above * and hence converges. Clearly the nonincreasing * sequence $\{F^n\zeta^+\}$ also converges.

Denote by $\overline{\zeta}$ a function with the properties $\overline{\zeta}(x,Z(x)) = 0$ and $\overline{\zeta}_z \leq 0$, and which is a fixed point of F:

$$F\overline{\zeta} = \overline{\zeta} = MP(\tau)S(\tau)\overline{\zeta} \quad . \tag{3.73}$$

There is at most one such $\overline{\zeta}$, since if there were two, $\overline{\zeta}_1$ and $\overline{\zeta}_2$, we would have

$$N_1(\overline{\zeta}_1 - \overline{\zeta}_2) = N_1(MP(\tau)S(\tau)\overline{\zeta}_1 - MP(\tau)S(\tau)\overline{\zeta}_2) \leq N_1(S(\tau)\overline{\zeta}_1 - S(\tau)\overline{\zeta}_2). \tag{3.74}$$

However, from the definitions of $S(\tau)$ and N_1 in (3.50) and (3.52a), respectively, one sees that, as long as $\partial\mathcal{D}_0 \cup \partial\mathcal{D}_R \neq \emptyset$, there is a constant $c(\tau) > 0$ such that

$$N_1(S(\tau)u_1 - S(\tau)u_2) \leq e^{-\tau c(\tau)}N_1(u_1 - u_2) \quad . \tag{3.75}$$

Inserting this into (3.74), we get that $N_1(\overline{\zeta}_1 - \overline{\zeta}_2) = 0$, or $\overline{\zeta}_1 = \overline{\zeta}_2$. This is not an assertion of uniqueness of the steady state solution of the actual physical problem, but of uniqueness of the approximate steady state solution generated by (3.68).

Remark 2.1 mentions the problem of uniqueness of the steady state solution of the physical dam problem. (Elementary analysis shows that, for some characteristic dimension a of $\mathcal{D}$,

$$c(\tau) = e^{-a^2/\tau} \tag{3.76}$$

will work. However, this is a very conservative estimate. The pseudo "time-dependent" problem of (3.46) and (3.47), the solution of which is approximated by (3.68), is different from the actual time-dependent problem discussed in the last section. However, as $\zeta(t)$ in (3.47) begins to converge to $\overline{\zeta}$, supp $\zeta(t)$ will vary slowly, and the operator F in (3.68b) will look at late times like the operator

$$\tilde{F}u \equiv X_{\text{supp } \zeta(t)} S(\tau)u \tag{3.77a}$$

where

$$(X_E u)(x,z) \equiv \int_z^{Z(x)} \chi_E(x,z')(-u_z(x,z'))dz' \tag{3.77b}$$

for any set $E \subset R^{N-1} \times (0,\infty)$. Repeated application of the operator $\tilde{F}$ generates a function $\tilde{\zeta}$ such that

$$\Delta(-\tilde{\zeta}_z) = \Delta\theta_z \quad , \quad x \in \text{supp } \zeta(t) \quad , \tag{3.78}$$

(Ref. 10), and thus for p given by (3.1), $\Delta p = 0$ in supp $\zeta(t)$. Hence, at late time, the free boundary does evolve in a manner similar to the evolution of the free boundary in the real time-dependent problem. In such a case, by remark 2.1, we may expect an exponential convergence of ζ^n to $\overline{\zeta}$ as $n \to \infty$, for the case when all components of supp $\overline{\zeta}$ are connected to $\partial\mathcal{D}_0 \cup \partial\mathcal{D}_R$, and we take a nondecreasing * sequence to get to $\overline{\zeta}$.)

We can do the split-step scheme (3.68a) generating ζ^{n+1} from ζ_0 in reverse order to get

$$\zeta'^{n+1} \equiv (S(\tau)MP(\tau))^{n+1}\zeta_0 \quad , \tag{3.79}$$

which converges to $\overline{\zeta}'$ as $n \to \infty$. It is clear that

$$\overline{\zeta}' = S(\tau)\overline{\zeta} \quad , \quad \overline{\zeta} = MP(\tau)\overline{\zeta}' \quad , \tag{3.80}$$

and thus

$$\overline{\zeta}' - \tau \leq \overline{\zeta} \leq \overline{\zeta}' \quad . \tag{3.81}$$

In the relatively uninteresting case (3.54) where the operator $S(t)$ is monotone, we can show that the true solution ζ satisfies

$$\overline{\zeta} \leq \zeta \leq \overline{\zeta}' \quad ,$$

and thus use (3.81) to obtain an error estimate. This sort of estimate was given already in (1.11) for algorithm II, and we have essentially reproduced its derivation (Ref. 3). Unfortunately, for the general case not covered by (3.54) we do not generally expect $\overline{\zeta}'$ to be an upper bound for ζ. Indeed, we would anticipate that $\overline{\zeta}' < \zeta$ at points in D near to the interior of $T \cap \partial D_R$.

We have not at this time produced an error estimate for the difference between $\overline{\zeta}$ and ζ, the solution of (3.45). The following steps indicate how we would proceed to get a precise estimate.

Step 1. We observe that (3.73) can be written, with $\overline{\Omega} \equiv \text{supp}\ \overline{\zeta} \cap D$, as

$$\overline{\zeta} = X_{\overline{\Omega}} S(\tau)\overline{\zeta} \quad . \tag{3.82}$$

Also, we note that, given regions Ω_1 and $\Omega_2 \subset \Omega_1$, the solutions of

$$\zeta_1 = X_{\Omega_1} S(\tau)\zeta_1 \quad , \quad \zeta_2 = X_{\Omega_2} S(\tau)\zeta_2 \quad , \qquad (3.83)$$

satisfy, for $\Delta\theta_z \leq 0$,

$$\zeta_1 \overset{*}{\geq} \zeta_2 \quad . \qquad (3.84)$$

Step 2. A check of (3.82) and (3.78) shows that $\overline{\zeta}_z$ satisfies the correct differential equation in $\overline{\Omega}$. Thus, the main task involves checking to what extent the boundary conditions on ζ are satisfied by $\overline{\zeta}$. Denoting by $\overline{T}$ the set of points on the top of $\partial\overline{\Omega}$,

$$\overline{T} \equiv \left\{ (x,z) \in C\ell(\overline{\Omega}) \,|\, z = \sup_{(x,z') \in \overline{\Omega}} z' \right\} \quad , \qquad (3.85)$$

we have no difficulty in verifying the condition

$$\overline{\zeta}(x,z) = 0 \quad , \quad (x,z) \in \overline{T} \quad , \qquad (3.86)$$

from (3.73). The error with which the other boundary conditions hold on parts of $\partial\overline{\Omega}$ near ∂D may be ascertained without too much difficulty. The only problems arise on parts of $\partial\overline{\Omega}$ which remain uniformly removed from ∂D in the limit $\tau \to 0$. We label these parts of $\partial\overline{\Omega}$ as $\partial\overline{\Omega}_f$.

Step 3. Since we do not have any *a priori* estimate of the regularity of $\partial\overline{\Omega}_f$, we may consider a modified algorithm which always generates a "smooth" boundary. The deviation of the solution generated by this modified algorithm from $\overline{\zeta}$ can be bounded by using a variation on the monotonicity arguments and operators

introduced by Brezis, Berger, and Rogers (Ref. 6) to study error bounds for approximate solutions of the Stefan problem. We let

$$(U_\delta^+ u)(x,z) \equiv \int_z^{Z(x)} \sup_{|y - (x,z')| \le \delta} (-u_z(y))dz' \quad ,(3.87a)$$

and

$$(U_\delta^- u)(x,z) \equiv \int_z^{Z(x)} \inf_{|y - (x,z')| \le \delta} (-u_z(y))dz' \quad .(3.87b)$$

For a set $E \subset D$, we define

$$U_\delta^\pm(E) \equiv \left\{(x,z)\Big|\left(U_\delta^\pm\left(\int_z^{Z(x)} \chi(E)dz'\right)\right)_z = -1\right\} \quad . \quad (3.88)$$

If ∂D has curvatures bounded in magnitude by $1/\delta$, the set $U_\delta^- U_\delta^+ E$ is the complement of the set of points in the complement of E through which the surface of a closed ball of radius δ can be passed in such a way that the whole ball lies outside E. The boundary of $U_\delta^- U_\delta^+ E$ is generated by rolling a "marble" of radius δ over the boundary of E, keeping the marble outside E.

When $u_z \le 0$ and $u(x,Z(x)) = 0$, we have

$$MP(\tau)U_\delta^+ u \mathrel{\underset{\sim}{>}^*} U_\delta^+ \chi_{U_\delta^+\{(x,z)|u \geq \tau\}} u \mathrel{\underset{\sim}{>}^*} U_\delta^+ \chi_{U_\delta^- U_\delta^+\{(x,z)|u \geq \tau\}} u \mathrel{\underset{\sim}{>}^*} U_\delta^+ MP(\tau)u \quad . \quad (3.89)$$

One may consider, in place of (3.68), the modified algorithm

$$\zeta_\chi{}^n = F_\chi{}^n \zeta_0 \quad , \quad (3.90a)$$

where

$$F_\chi \zeta \equiv \chi_{U_\delta^- U_\delta^+\{(x,z)|S(\tau)\zeta \geq \tau\}} S(\tau)\zeta \quad . \quad (3.90b)$$

As $n \to \infty$, $\zeta_\chi{}^n$ will approach a "steady state" $\overline{\zeta}_\chi$ satisfying

$$\overline{\zeta}_\chi = \chi_{U_\delta^- U_\delta^+\{(x,z)|S(\tau)\overline{\zeta}_\chi \geq \tau\}} S(\tau)\overline{\zeta}_\chi \quad . \quad (3.91)$$

It is clear from (3.84) that

$$\bar{\zeta}_X \geq^* \bar{\zeta} \quad , \tag{3.92}$$

and moreover $\bar{\Omega}_X = \text{supp}\ \bar{\zeta}_X \cap D$ has the property that parts of $\partial\bar{\Omega}_X$ which project into $\bar{\Omega}_X$ will be smooth.

To bound $N_1(\bar{\zeta}_X - \bar{\zeta})$, we may formulate a third algorithm, according to which a quantity $(\zeta_\delta^+)^n$ is constructed as

$$(\zeta_\delta^+)^n = (F_\delta^+)^n \zeta_0 \quad , \tag{3.93a}$$

where

$$F_\delta^+ \zeta = MP(\tau)(S(\tau)_\delta^+)\zeta \tag{3.93b}$$

and the operator $S(\tau)_\delta^+$ is required to satisfy

$$S(\tau)_\delta^+ U_\delta^+ \zeta \geq^* U_\delta^+ S(\tau)\zeta \quad . \tag{3.93c}$$

Then, upon setting $\zeta_0 \equiv 0$ in (3.68), (3.90), and (3.93), we obtain for the steady state $\bar{\zeta}_\delta^+$ of (3.93),

$$\bar{\zeta}_\delta^+ \geq^* \bar{\zeta}_X \geq^* \bar{\zeta} \quad , \tag{3.94}$$

and we can bound $N_1(\bar{\zeta}_X - \bar{\zeta})$ by $N_1(\bar{\zeta}_\delta^+ - \bar{\zeta})$.

We may find an appropriate operator $S(\tau)_\delta^+$ by replacing $\Delta\theta_Z$ and $S_0(t)$ as they occur in (3.50) and (3.48) by $(\Delta\theta_Z)_\delta^+$ and $S_0(t)_\delta^+$, respectively. (For example, in the case where $\partial D_I = \emptyset$, we may let $D \to U_\delta^+ D$ in the definition of $S_0(t)_\delta^+$ and we may let $(\Delta\theta)_\delta^+ = U_\delta^+ \Delta\theta$.) Given $S_0(\tau)_\delta^+$ and $(\Delta\theta_Z)_\delta^+$, one can use, for $u_2 \geq^* 0$,

$$MP(\tau)(u_1 + u_2) \leq^* MP(\tau)u_1 + u_2 \quad , \tag{3.95}$$

to get

$$(F_\delta^+)^{n+1}(0) - F^{n+1}(0) \leq^* \int_z^{Z(x)} \int_0^{(n+1)\tau} [S_0(t)_\delta^+(-\Delta\theta_z)_\delta^+ - S_0(t)(-\Delta\theta_z)]dtdz'$$

$$\leq^* \int_z^{Z(x)} \int_0^{\infty} [S_0(t)_\delta^+(-\Delta\theta_z)_\delta^+ - S_0(t)(-\Delta\theta_z)]dtdz' \quad . \tag{3.96}$$

This last bound is seen to be an upper * bound on $\overline{\zeta}_\delta^+ - \overline{\zeta} \geq^* 0$.

It may be evaluated directly by observing from (3.48) that

$$\Psi = \int_0^\infty S_0(t)dt\psi$$

satisfies (if $\partial D_0 \cup \partial D_R \neq \emptyset$)

$$\Delta\Psi = -\psi \quad , \quad x \in D \quad , \tag{3.97a}$$

$$\Psi = 0 \quad , \quad x \in \partial D_0 \cup \partial D_R \quad , \tag{3.97b}$$

$$\frac{\partial\Psi}{\partial n} = 0 \quad , \quad x \in \partial D_I \quad . \tag{3.97c}$$

Finally, one may obtain

$$N_1(\overline{\zeta}_\delta^+ - \overline{\zeta}) \leq \int_{\widetilde{D}} [-(\overline{\zeta}_\delta^+)_z + \overline{\zeta}_z]dxdz \quad , \tag{3.98}$$

where $\widetilde{D}$ is a suitable region for which the integral can be bounded and which we know *a priori* to contain supp $\overline{\zeta}_\delta^+$.

Step 4. It is obvious from (3.91) that $\overline{\zeta}_x = 0$ on the set

$$\mathbf{T}_x \equiv \left\{(x,z) \in C\ell(\overline{\Omega}_x) \,\middle|\, z = \sup_{(x,z') \in \overline{\Omega}_x} z'\right\} \quad . \tag{3.99}$$

It can also be shown (Ref. 10) that for given δ, and $\tau \to 0$,

$$(\overline{\zeta}_x)_z = O\left(\left(\tau \ln \frac{1}{\tau}\right)^{\frac{1}{2}}\right) \text{ on } \{(x,z) \in \mathbf{T}_x \,|\, \mathrm{dist}((x,z), \partial D_R \cup \partial D_I) \geq \delta\} \quad . \tag{3.100}$$

Regarding the free boundary condition, that $\Delta\overline{\zeta}_x \to 1$ just below the set $\{(x,z) \in \mathbf{T}_x \,|\, \mathrm{dist}((x,z), \partial D) \geq \delta\}$, that will follow, in the limit as $\tau \to 0$, directly from (3.91).

Our guess is that careful handling of all the estimates will finally yield, for the N_1 norm of the difference between $\overline{\zeta}$ and ζ, a result which is $O\left(\left(\tau \ln \frac{1}{\tau}\right)^{1/2}\right)$.

To this point we have assumed that $\partial\mathcal{D}_{I_1} = \emptyset$. Let us now see how we may modify the algorithm (3.68) when $\partial\mathcal{D}_{I_1} \neq \emptyset$. The following scheme may be used.

$S_0(t)$ is still given by (3.48). Let $G_0(t,x,z,\tilde{x},\tilde{z})$ be the derivative of $(S_0(t)\xi_0)(x,z)$ with respect to $\xi_0(\tilde{x},\tilde{z})$:

$$(S_0(t)\xi_0)(x,z) = \int_{\mathcal{D}} G_0(t,x,z,\tilde{x},\tilde{z})\xi_0(\tilde{x},\tilde{z})d\tilde{x}d\tilde{z} \quad . \quad (3.101)$$

We next introduce, in place of $S(t)$ in (3.50), the operator

$$\begin{aligned}(S^*(t)\zeta_0)(x,z) \equiv \int_z^{Z(x)} \Bigg\{ & (S_0(t)(-\zeta_{0z}))(x,z') \\ & - \left(\left(\int_0^t S_0(t')dt'\right)\Delta\theta_z\right)(x,z') \\ & - \int_{\partial\mathcal{D}_{I_1}} \left(\int_0^t G_0(t',x,z',\tilde{x},\tilde{z})dt'\right)\vec{k} \cdot n(\tilde{x},\tilde{z})d\tilde{S}\Bigg\} dz' \ .\end{aligned}$$
$$(3.102)$$

$S^*(\tau)$ is the operator that would be appropriate in the computational algorithm if $\partial\Omega \supset \partial\mathcal{D}_{I_1}$. In general this is not the case, and it does not generally follow from $\zeta_0 \geq^* 0$ that $S^*(\tau)\zeta_0 \geq^* 0$. Since we want the iterative solutions we generate to satisfy $\zeta^n \geq^* 0$ (nonnegative pressure), we shall replace (3.68) in the case $\partial\mathcal{D}_{I_1} \neq \emptyset$ by

$$\zeta^{*n} = F^{*n}\zeta_0 \quad , \quad (3.103a)$$

where

$$F^*\zeta \equiv MP(\tau)M^*S^*(\tau)\zeta \tag{3.103b}$$

and

$$M^*\zeta \equiv \max{}^*(\zeta,0) \quad . \tag{3.103c}$$

Remark 3.4: The various algorithms (3.68), (3.90), (3.93), and (3.103) may be replaced by algorithms where in place of $S_0(t)$ given by (3.48), we work solely with semi-groups for the diffusion equation in $R^{N-1}x(0,\infty)$. Operations with such semi-groups are interspersed with multiplication by operators X_E for given sets E, defined in (3.77b), to approximately satisfy the Dirichlet conditions on ∂D, and also with multiplication by suitable reflection operators, to approximately satisfy the Neumann conditions on the remainder of ∂D.

For the sake of mathematical completeness, we will discuss the case where the set $\{z_f(x)\}$ defined in (3.8) may contain more than one element for a given x, and where the condition (3.11) need not hold. The essential difference in our approach will involve replacing the operator $P(\tau)$ in (3.103) by something more complicated.

Given quantities $z_1(\cdot)$, $z_2(\cdot)$, and a function $u(\cdot,z)$, we define, for $z_1(\cdot) > z_2(\cdot) \geq 0$,

$$\mu(z_1(\cdot),z_2(\cdot),z;u(\cdot,z)) \equiv \begin{cases} 0 & z \geq z_1(\cdot) \\ \min(u(\cdot,z) - u(\cdot,z_1(\cdot)),\tau) & z_1(\cdot) \geq z \geq z_2(\cdot) \\ \min(u(\cdot,z_2(\cdot)) - u(\cdot,z_1(\cdot)),\tau) & z_2(\cdot) \geq z \geq 0 \end{cases} \quad . \tag{3.104}$$

We recall that we defined the sets of integers $S_D(x)$ in (3.10a) and (3.10b), according to whether $n(x)$ was odd or even.

Let the operator $Q(\tau,x)$ be defined by

$$(Q(\tau,x)u)(x,z) \equiv u(x,z) - \mu(\infty,z_1(x),z;u(x,z)) - \sum_{i\in S_D(x)} \mu(z_{2i}(x),z_{2i+1}(x),z;u(x,z)) \tag{3.105a}$$

for $Z(x) = \infty$, and by

$$(Q(\tau,x)u)(x,z) \equiv - \sum_{i\in S_D(x)} \mu(z_{2i-1}(x),z_{2i}(x),z;u(x,z)) + u(x,z) \tag{3.105b}$$

when $Z(x) < \infty$. Finally, in the general case we replace (3.103) by

$$\zeta_Q{}^n = F_Q{}^n \zeta_0 \quad , \tag{3.106a}$$

where

$$F_Q \zeta \equiv Q(\tau,x)M^* S^*(\tau)\zeta \quad . \tag{3.106b}$$

4. A Different Problem

We shall consider a variation on the problems discussed heretofore, but only for the case $\partial D_2 = \emptyset$, where ∂D_2 is defined by (1.20) and (1.21). Suppose the boundary values of p, prescribed as p_R on ∂D in accordance with (1.13d), are increased to

$$\tilde{p} = p_R \chi(\partial D_R) + p' \quad , \quad (x,z) \in \partial D_0 \cup \partial D_R \quad , \tag{4.1a}$$

$$p' \geq 0 \quad , \quad \{(x,z) \in \partial D | p' > 0\} \equiv \Gamma \subset \partial D_0 \cup \partial D_R \quad . \tag{4.1b}$$

If we restrict our attention to components of Ω such that $\partial\Omega \cap (\partial D_0 \cup \partial D_R) \neq \emptyset$, we obtain

$$\tilde{\Omega} \supset \Omega \quad , \quad p' \equiv \tilde{p} - p \geq 0 \text{ on } \partial\Omega_f \quad . \tag{4.2}$$

On the other hand,

$$p' = 0 \quad , \quad (x,z) \in (\partial D_0 \cup \partial D_R) \cap \partial\Omega - \Gamma \quad , \tag{4.3a}$$

$$\frac{\partial p'}{\partial n} = 0 \quad , \quad (x,z) \in \partial D_I \cap \partial\Omega \quad . \tag{4.3b}$$

We get $p' \geq 0$ on $\partial D_I \cap \partial\Omega$, since otherwise there would be a violation of (4.3b), and thus from (4.1)-(4.3),

$$\int_{(\partial D_0 \cup \partial D_R) \cap \partial\Omega - \Gamma} \frac{\partial p'}{\partial n} dS \leq 0 \quad . \tag{4.4}$$

According to (1.31a), $\phi = -p - z$, $\tilde{\phi} = -\tilde{p} - z$, and

$$\phi' = \tilde{\phi} - \phi = -p' \quad , \quad x \in \Omega \quad . \tag{4.5}$$

Thus

$$\int_{(\partial D_0 \cup \partial D_R) \cap \partial\Omega - \Gamma} \frac{\partial \phi'}{\partial n} dS \geq 0 \quad . \tag{4.6}$$

Since $\frac{\partial \phi'}{\partial n} = 0$ for $(x,z) \in \partial D_I \cap \partial\Omega$, (4.6) implies

$$\int_{\partial D \cap \partial\Omega - \Gamma} \frac{\partial \phi'}{\partial n} dS \geq 0 \quad . \tag{4.7}$$

Now we use the facts that $\Delta\phi' = 0$ in Ω and $\frac{\partial \phi}{\partial n} = 0$ on $\partial\Omega_f$:

$$\int_{\Gamma \cap \partial\Omega} \frac{\partial \phi'}{\partial n} dS + \int_{\partial\Omega_f} \frac{\partial \tilde{\phi}}{\partial n} dS \leq 0 \quad . \tag{4.8}$$

In $\tilde{\Omega} - \Omega$ we have $\Delta\tilde{\phi} = 0$ and the boundary condition $\frac{\partial \tilde{\phi}}{\partial n} = 0$ on $\partial\tilde{\Omega}_f$ leads to

$$\int_{\partial\Omega_f} \frac{\partial \tilde{\phi}}{\partial n} dS = \int_{\partial(\tilde{\Omega}-\Omega) \cap \partial D} \frac{\partial \tilde{\phi}}{\partial n} dS \quad . \tag{4.9}$$

Because of the condition (1.31j), the assumption $\partial D_2 = \emptyset$, and the fact that $\frac{\partial \tilde{\phi}}{\partial n} = 0$ on $\partial(\tilde{\Omega} - \Omega) \cap \partial D_I$, we get

$$\int_{\partial(\tilde{\Omega}-\Omega) \cap \partial D - \Gamma} \frac{\partial \tilde{\phi}}{\partial n} dS \geq 0 \quad . \tag{4.10}$$

Thus one derives from (4.8)-(4.10) that

$$\int_{\Gamma\cap\partial\Omega} \frac{\partial\phi'}{\partial n} dS + \int_{\partial(\tilde{\Omega}-\Omega)\cap\Gamma} \frac{\partial\tilde{\phi}}{\partial n} dS \leq 0 \quad . \tag{4.11}$$

In physical language, (4.11) establishes the monotone increasing dependence of the flow into the dam across Γ on the pressures prescribed on Γ. Now we may imagine a problem in which Γ is the boundary of D with a reservoir, whose height is not known but across which a total flux is prescribed. Part of the problem is to determine the height of the reservoir. If one establishes upper and lower limits for the reservoir height, such that at the upper limit the flow out of the reservoir will be too large and at the lower limit the flow out will be too small, then the monotonicity just deduced establishes the existence of a unique reservoir height for which the flux condition is satisfied.

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